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PRESERVATION OF PRODUCTS BY FUNCTORS CLOSE TO REFLECTORS

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It is shown that reflectors and similar functors in algebraic and topological-algebraic structures in many cases commute with products. In particular, reflectors of the category of (semi) topological semigroups into the subcategory of compact topological semigroups or groups have this property. The proofs are straightforward and avoid the use of almost periodic functions.

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1. Introduction

In this paper we study functors close to reflectors and we consider the question in which cases they preserve products. It turns out that this is often the case when some kind of algebraic structure is involved. Our interest in this problem was stimulated by the fact that we did not understand the proof in [11] that products of topological groups are preserved by the Bohr compactification functor (it is all right if all groups involved are abelian). All later papers dealing with this question known to us are based on the theory of almost periodic functions. Our approach is directly based on the categorical properties involved and it applies to many other situations.

Let $F:\mathcal{K}_1\to\mathcal{K}_2$ be a covariant functor between categories $\mathcal{K}_1$ and $\mathcal{K}_2$ (for categorical notions we refer to [10]) and assume that for a set $\{X_i\}$ of objects in $\mathcal{K}_1$ both the products $\prod X_i$ in $\mathcal{K}_1$ and $\prod FX_i$ in $\mathcal{K}_2$ exist. Then there is a canonical morphism $\mu_{\{X_i\}}:F(\prod X_i)\to\prod FX_i$ (shortly: μ), defined uniquely by the condition that the following diagram commutes for every j.

$$\begin{tikzcd}
F(\Pi X_i) \arrow[rr, "\mu"] \arrow[dr, "F(pr_{X_j})"'] & & \Pi FX_i \arrow[dl, "pr_{FX_j}"] \\
& FX_j &
\end{tikzcd}$$

(here *pr* means: projection). If μ is an isomorphism in $\mathcal{K}_2$ then we shall say that 'F preserves the product of $\{X_i\}$' or 'F commutes with the product of $\{X_i\}$'. There are many examples where F always preserves products, e.g. if F is a right adjoint, or if F is a covariant Hom-functor, or if F is a product functor. As we are more interested in reflectors, these results are of little use for us (see however the beginning of Section 2). We shall consider a situation which always occurs if F is a reflector, but which is more general: we shall assume that $\mathcal{K}_1 = \mathcal{K}_2 =: \mathcal{K}$, so that F is an endofunctor of K, and we shall assume that there is a natural transformation $\eta: 1_{\mathcal{K}} \to F$. In that case one has the equality

$$\mu_{\{X_i\}} \circ \eta_{\prod X_i} = \prod \eta_{X_i} \tag{1}$$

which follows from the following commutative diagram:

$$\begin{tikzcd}
\Pi X_i \arrow[rr, "\eta_{\Pi X_i}"] \arrow[dr, "\Pi \eta_{X_i}"] \arrow[dd, "pr_{X_j}"'] & & F\Pi X_i \arrow[dl, "\mu"'] \arrow[dd, "Fpr_{X_j}"] \\
& \Pi FX_i \arrow[dr, "pr_{FX_j}"'] & \\
X_j \arrow[rr, "\eta_{X_j}"'] & & FX_j
\end{tikzcd}$$

In the sequel we shall sometimes say that such an F is 'close to a reflector'. Let us now summarize several relevant known results from various structures.

Examples. (1) In the category of topological spaces one of the most studied reflections is the Čech–Stone compactification. It is known [8] that for completely regular spaces μ is a homeomorphism if and only if $\prod X_i$ is pseudocompact (granted some non-trivality condition). A similar assertion is true for zero-dimensional spaces and the Banaschewski compactification [13, 3]. The problem when the Hewitt real compactification υ preserves products is still open. For partial results see e.g. [5], [12] and [24], where one can find other references. In any case, the property $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is not a topological property of the space $X \times Y$ (see [12]). In the positive results for υ (and, similarly, in results for the topological completion; see [26]), local compactness plays an important role. This is not by accident: locally compact spaces are so-called exponential objects (i.e.—$\times X$ has a right adjoint) and in [28] for such objects X situations are characterized where $F(X \times Y) = FX \times FY$ (one can find in [28] other references to similar results, e.g. by B. Day and O. Wyler).

(2) In the previous example the failure of μ to be an isomorphism (or even an injection) in general is basically due to the fact that a dense embedding of a space X into a space of the form FY is not uniquely determined by X. But this is really the case for completions of structures with uniformly continuous mappings as in **Metr, Unif, TopVs, Norm**: completions are unique, hence the completion functor preserves all products.

(3) An interesting example is Herrlich's wild reflection of **Top** into the full subcategory generated by $\{C^\kappa \mid \kappa$ a cardinal number$\}$, where C is a strongly rigid Hausdorff space. This subcategory is reflective in **Haus** (see [9]) and a straightforward argument shows that the reflector preserves a product $\prod X_i$ iff every continuous mapping $\prod X_i \to C$ depends on at most one coordinate.

(4) Another type of reflections are those where the units are just bijections or surjections. As is well-known (and easy to prove), the T_0-modification in **Top** preserves finite products; see also Application 2 ahead. The T_1-, T_2- and T_3-modifications do not preserve all finite products (this is also well-known; as an example, consider $1 \times \eta : Q \times X \to Q \times Y$, where $X = \omega \times (\omega + 1)$ with the topology in which all points of $\omega \times \omega$ are discrete, while a point $(n, \omega) \in \omega \times (\omega + 1)$ has a nbd base consisting of sets of the form $\{(i, k) \mid i \leq n, m \leq k \leq \omega\}$ for some $m \in w$; the space Y is the quotient of X obtained by identification of the subset $\{(n, \omega) \mid n \in \omega\}$ to one point; the quotient map $\eta : X \to Y$ is the T_2- (hence T_1-) modification of X). In certain situations, T_1-modifications preserve finite products, e.g. of symmetric spaces (i.e., spaces in which $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$ or of (not necessarily T_1-) completely regular spaces; but these are really instances of T_0-modifications. For the $T_{3\frac{1}{2}}$-modification F of (not necessarily T_1-) topological spaces probably the strongest results are in [15]: if X is completely regular then $F(X \times Y) = X \times FY$ for every regular space Y iff X is locally compact (cf. also the final remark in Example 1 above). In **Unif**, the precompact modification functor F is an example of a reflector where the units are not embeddings. It commutes with the product of $\{X_i\}$ if at most one of the spaces X_i is not precompact; moreover, for any space X, $F(X \times X) = FX \times FX$ iff X is precompact (see [4]).

(5) Let $\mathscr{K}$ be the category of partially ordered sets and monotone mappings which are either sup-preserving or inf-preserving or sup-inf-preserving. Then the reflection of $\mathscr{K}$ into the full subcategory of complete partially ordered sets preserves products (the form of the reflection depends on the type of morphisms; compare with [10, p. 180]).

(6) Let **SGrp** denote the category of semigroups; if not stated otherwise we shall assume that each semigroup has a unit and that homomorphisms of semigroups preserve the units. With **TopSGrp** (respectively, **STopSGrp**) we shall denote the category of all topological (respectively, semitopological) semigroups; recall that in a topological semigroup S the semigroup operation $S \times S \to S$ is simultaneously continuous, whereas in a semitopological semigroup it is only separately continuous. Apart from Holm's paper mentioned above the following papers deal with preservation of products by reflections of these categories into their full subcategories of

compact objects (in obvious notation, **CompSTopSGrp**, **CompTopSGrp** and **CompTopGrp** are reflective subcategories of **STopSGrp**; the reflections of an object X in these categories are often denoted as X^{WAP} (weakly almost periodic compactification), X^{AP} (almost periodic compactification) and X^{SAP} (strongly almost periodic compactification)): generalizing work of [19] and [2], [17] shows that the functor $(\cdot)^{\mathrm{AP}}$ preserves arbitrary products, and in [2], [16] it is shown that $(\cdot)^{\mathrm{WAP}}$ does not preserve finite products.

(7) Dierolf proved in [7] that every bireflection (i.e. the unit consists of bimorphisms) in the category **TopVS** of topological vector spaces preserves products. This was generalized in [29] for endofunctors F of productive subcategories $\mathscr{K}$ of **TopVS** for which there exists a certain natural bitransformation $\eta: 1_{\mathscr{K}} \to F$. Our results in Section 2 below are even more general.

(8) Let G be a topological group and let $\mathscr{K}$ be the category $\mathbf{Top}^G$ of all topological transformation groups with acting group G and continuous equivariant mappings (see e.g. [30]). Let for an object X of $\mathbf{Top}^G$, $\eta_X: X \to FX$ be the reflection of X into the subcategory of compact objects in $\mathbf{Top}^G$. In the same way as in Example 1, if G is locally compact and locally connected, then $\mu: F(\prod X_i) \to \prod FX_i$ is an isomorphism iff $\prod X_i$ is pseudocompact (apart from trivial cases); see [31, 32].

We shall present our results for the situation described in the beginning of this introduction in two parts: Section 2 deals with finite products and Section 3 with infinite products. Although in both cases the approach has a common idea, in details different procedures must be used. Also, for infinite products the results are less general. Also, in order to avoid intricate formulations we have refrained from writing down all results in the greatest possible generality. An inconvenient consequence is that at some places we have to refer to a proof rather than to the corresponding result.

2. Finite products

The main results of this section are stated for algebraic structures (with or without an additional topological structure). In most cases a functor close to a reflector preserves finite products. For non-algebraic structures the method gives a weaker version of preservation (e.g. μ a bijection but not necessarily an isomorphism), which is nevertheless useful.

As observed already in the Introduction, sometimes the preservation of (finite) products by reflections follows from general results. For example, let $\mathscr{K}$ be a category where finite products and coproducts exist and coincide (a so-called *semi-additive category*; see [10, Section 40]) and let $F: \mathscr{K} \to \mathscr{K}_1$ be a reflector into a full subcategory $\mathscr{K}_1$ of $\mathscr{K}$. Then F preserves coproducts, hence all finite products (in $\mathscr{K}_1$, products and coproducts coincide as well). Examples of semi-additive categories are **Ab**,

R-Mod (R any ring), their topological versions **TopAb**, **TopVS** and their full subcategories. Also the full subcategories of all commutative objects in **SGrp** and **TopSGrp** (*not* of **STopSGrp**) are semi-additive (together with Theorem 3 in Section 3 below this accounts e.g. for the preservation result in [19]). We shall consider a slightly more general situation: an endofunctor of a semi-additive category which is close to a full reflector. As a motivation for the following definition, we mention the following characterization: a category $\mathscr{K}$ is semi-additive iff it has finite products, it is 'pointed' (i.e., $\mathscr{K}(X, Y)$ contains a unique zero morphism $e_{X,Y}$ for any two objects X and Y in $\mathscr{K}$) and it has a 'categorical' binary operation ϕ. This last condition means the following: let $D_{\mathscr{K}}X := X \times X$ and $D_{\mathscr{K}}f := f \times f$ (X an object and f a morphism in $\mathscr{K}$); then ϕ: $D_{\mathscr{K}} \to 1_{\mathscr{K}}$ is a natural transformation such that for each object X in $\mathscr{K}$ the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "e_X \Delta 1_X"] \arrow[d, "1_X \Delta e_X"'] \arrow[dr, "1_X"] & X \times X \arrow[d, "\phi_X"] \\
X \times X \arrow[r, "\phi_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times X \times X \arrow[r, "1_X \times \phi_X"] \arrow[d, "\phi_X \times 1_X"'] & X \times X \arrow[d, "\phi_X"] \\
X \times X \arrow[r, "\phi_X"'] & X
\end{tikzcd}$$

Here $e_X := e_{X,X}$, the zero morphism of X, and Δ denotes the diagonal product operation. (That this characterization is equivalent with the definition of semi-additive category as given in [10] follows easily from the observation that if $\mathscr{K}$ is semi-additive, then one can take for ϕ_X the codiagonal map; conversely, if $\mathscr{K}$ satisfies the above conditions, then 'addition' of morphisms $f, g: X \to Y$ can be defined by $f+g := \phi_X \circ (f \times g) \circ \delta_X$ where δ_X is the diagonal map.) In the characterization above, the condition that ϕ is a natural transformation expresses two properties, namely that each ϕ_X is a morphism in $\mathscr{K}$ and that all morphism of $\mathscr{K}$ are homomorphisms with respect to ϕ. We shall now relax the first property, while keeping the second one. The reason is, that in non-commutative algebraic structures (e.g. for **Grp**) the binary operation $X \times X \to Y$ and, consequently, the canonical mapping $X \times Y \to X + Y$ are not morphisms in the category under consideration, but in some auxiliary 'underlying' category (e.g. **Set**). Therefore we introduce the following notion of 'relative' semi-additivity:

Definition. A category $\mathscr{K}$ is said to be *semi-additive over a category* $\mathscr{X}$ whenever it satisfies the following conditions:

(1) $\mathscr{K}$ has finite products;

(2) $\mathscr{K}$ has zero-morphisms (for objects X and Y, $e_{X,Y}$ will denote the zero-morphism from X to Y, and $e_X := e_{X,X}$);

(3) There is a faithful functor $|{-}|: \mathscr{K} \to \mathscr{X}$ which preserves all finite products and reflects all isomorphisms;

(4) There is a natural transformation $\phi: |D_{\mathscr{K}}| \to |{-}|$ such that

$$\phi \circ (|1|\Delta|e|) = \phi \circ (|e|\Delta|1|) = |1| \quad \text{and} \quad \phi \circ (\phi \times |1|) = \phi \circ (|1| \times \phi).$$

Here $|1|: X \mapsto |1_X| = 1_{|X|}$ and $|e|: X \mapsto |e_X|$ are natural transformations from $|-|$ to itself (of course $|D_{\mathscr{K}}| := |-| \circ D_{\mathscr{K}}$, a functor from $\mathscr{K}$ to $\mathscr{X}$). So for each object X of $\mathscr{K}$ there is a morphism $\phi_X : |X \times X| = |X| \times |X| \to |X|$ in $\mathscr{X}$ such that the above diagrams (with obvious modifications) commute.

The categories **Grp**, **SGrp** and their full subcategories are semi-additive over **Set**; the category **Rng** is semi-additive over **Ab**, over **SGrp** and over **Set**; **TopGrp**, **TopSGrp** and **TopRng** are semi-additive over **Top**. Similarly, the categories of uniform groups or convergence groups are semi-additive over the category of uniform spaces or convergence spaces, respectively. The categories of semi-topological structures (i.e. $\phi_X : X \times X \to X$ is separately continuous: e.g. **STopGrp**, **STopSGrp**, etc.) are *not* semi-additive over some category: they are not so over **Top** (because ϕ_X is not continuous) and they are not so over any other category (**Set**, for example) because then $|-|$ does not reflect isomorphisms (of course, we could leave this condition out of the definition, but then we would have to include it in Theorem 1 below).

In the following theorem, $F(\mathscr{K})$ and $|F(\mathscr{K})|$ denote the subcategories of $\mathscr{K}$ and $\mathscr{X}$, respectively, generated by the objects $F(X)$ and $|F(X)|$ with X in $\mathscr{K}$.

Theorem 1. *Let the category $\mathscr{K}$ be semi-additive over the category $\mathscr{X}$, and let $F : \mathscr{K} \to \mathscr{K}$ be a covariant functor. If there is a natural transformation $\eta : 1_{\mathscr{K}} \to F$ such that, for each object X of $\mathscr{K}$, $|\eta_X|$ is an epimorphisms with respect to $|F(\mathscr{K})|$, then F preserves finite products.*

Proof. First observe that the functor F preserves zero-morphisms: for any pair X, Y of objects in $\mathscr{K}$ the equalities

$$|e_{FX,FY}| \circ |\eta_X| = |e_{X,FY}| = |\eta_Y \circ e_{X,Y}|$$

hold (compositions with zero-morphisms are again zero-morphisms), as well as

$$|F(e_{X,Y})| \circ |\eta_X| = |\eta_Y \circ e_{X,Y}|$$

(η is a natural transformation). Since $|e_{FX,FY}|$ and $|F(e_{X,Y})|$ are morphisms in $|F(\mathscr{K})|$, the epi-property of η implies that $|F(e_{X,Y})| = |e_{FX,FY}|$. Because $|-|$ is faithful, it follows that $F(e_{X,Y}) = e_{FX,FY}$ for all objects X, Y in $\mathscr{K}$.

In order to show that $\mu : F(X \times Y) \to FX \times FY$ is an isomorphism in $\mathscr{K}$ it is sufficient to show that $|\mu|$ is an isomorphism in $\mathscr{X}$. We shall show that its inverse in $\mathscr{X}$ is the morphism

$$\nu := \phi_{F(X \times Y)} \circ |\{F(1_X \Delta e_{X,Y}) \circ \mathrm{pr}_{FX}\} \Delta \{F(e_{Y,X} \Delta 1_Y) \circ \mathrm{pr}_{FY}\}|$$

(X, Y objects in $\mathscr{K}$). For convenience, we shall omit in the remainder of the proof all occurrences of the functor $|-|$, understanding the intention to consider all morphisms as belonging to the category $\mathscr{X}$.

To prove $\mu \circ \nu = 1_{FX \times FY}$ is equivalent with showing that $\mathrm{pr}_{FX} \circ \mu \circ \nu = \mathrm{pr}_{FX}$ and $\mathrm{pr}_{FY} \circ \mu \circ \nu = \mathrm{pr}_{FY}$. We shall prove the first of these equalities:

$$\begin{aligned}
\mathrm{pr}_{FX} \circ \mu \circ \nu &\overset{(1)}{=} \phi_{FX} \circ [F(\mathrm{pr}_X) \times F(\mathrm{pr}_X)] \\
&\quad \circ \{[F(1_X \Delta e_{X,Y}) \circ \mathrm{pr}_{FX}] \Delta [F(e_{Y,X} \Delta 1_Y) \circ \mathrm{pr}_{FY}]\} \\
&= \phi_{FX} \circ \{[F(\mathrm{pr}_X) \circ F(1_X \Delta e_{X,Y}) \circ \mathrm{pr}_{FX}] \Delta [F(\mathrm{pr}_X) \circ F(e_{Y,X} \Delta 1_Y) \circ \mathrm{pr}_{FY}]\} \\
&\overset{(2)}{=} \phi_{FX} \circ \{\mathrm{pr}_{FX} \Delta e_{FX \times FY, FX}\} \\
&= \phi_{FX} \circ (\mathrm{pr}_{FX} \times \mathrm{pr}_{FX}) \circ \{1_{FX \times FY} \Delta e_{FX \times FY}\} \\
&\overset{(3)}{=} \mathrm{pr}_{FX} \circ \phi_{FX \times FY} \circ (1_{FX \times FY} \Delta e_{FX \times FY}) \overset{(4)}{=} \mathrm{pr}_{FX}.
\end{aligned}$$

Here equality (1) is based on the fact that $F(\mathrm{pr}_X) \circ \phi_{F(X \times Y)} = \phi_{FX} \circ (F(\mathrm{pr}_X) \times F(\mathrm{pr}_X))$ which follows from ϕ being a natural transformation; note also that $\mathrm{pr}_{FX} \circ \mu = F(\mathrm{pr}_X)$. Equality (3) follows similarly from ϕ being a natural transformation. In (2) it is used that $F(\mathrm{pr}_X \circ (1_X \Delta e_{X,Y})) = F(1_X) = 1_{FX}$ and $F(\mathrm{pr}_X \circ (e_{Y,X} \Delta 1_Y)) = F(e_{Y,X}) = e_{FY,FX}$. Finally, (4) uses one of the axioms of ϕ.

Next we show that $\nu \circ \mu = 1_{F(X \times Y)}$ or equivalently (by the assumption on η), $\nu \circ \mu \circ \eta_{X \times Y} = \eta_{X \times Y}$. Since $\mu \circ \eta_{X \times Y} = \eta_X \times \eta_Y$ we must prove $\nu \circ (\eta_X \times \eta_Y) = \eta_{X \times Y}$; as follows:

$$\begin{aligned}
\nu \circ (\eta_X \times \eta_Y) &\overset{(5)}{=} \phi_{F(X \times Y)} \circ \{[F(1_X \Delta e_{X,Y}) \circ \eta_X \circ \mathrm{pr}_X] \Delta [F(e_{Y,X} \Delta 1_Y) \circ \eta_Y \circ \mathrm{pr}_Y]\} \\
&= \phi_{F(X \times Y)} \circ \{[\eta_{X \times Y} \circ (1_X \Delta e_{X,Y}) \circ \mathrm{pr}_X] \Delta [\eta_{X \times Y} \circ (e_{Y,X} \Delta 1_Y) \circ \mathrm{pr}_Y]\} \\
&= \phi_{F(X \times Y)} \circ (\eta_{X \times Y} \times \eta_{X \times Y}) \circ \{(1_X \times e_Y) \Delta (e_X \times 1_Y)\} \\
&= \eta_{X \times Y} \circ \phi_{X \times Y} \circ \{(1_X \times e_Y) \Delta (e_X \times 1_Y)\} \overset{(6)}{=} \eta_{X \times Y}.
\end{aligned}$$

Here properties of ϕ and η as natural transformations are used. Also, (5) requires the definition of ν and the equality $\mathrm{pr}_{FX} \circ (\eta_X \times \eta_Y) = \eta_X \circ \mathrm{pr}_X$ (similarly for pr_{FY}), and (6) follows from the equality

$$\phi_{X \times Y} \circ \{(1_X \times e_Y) \Delta (e_X \times 1_Y)\} = 1_{X \times Y},$$

which can be proved by composing both sides with pr_X and pr_Y:

$$\begin{aligned}
&\mathrm{pr}_X \circ \phi_{X \times Y} \circ \{(1_X \times e_Y) \Delta (e_X \times 1_Y)\} \\
&\quad = \phi_X \circ (\mathrm{pr}_X \times \mathrm{pr}_X) \circ \{(1_X \times e_Y) \Delta (e_X \times 1_Y)\} \\
&\quad = \phi_X \circ \{(\mathrm{pr}_X \circ (1_X \times e_Y)] \Delta [\mathrm{pr}_X \circ (e_X \times 1_Y)]\} \\
&\quad = \phi_X \circ \{(1_X \circ \mathrm{pr}_X) \Delta (e_X \circ \mathrm{pr}_X)\} = \phi_X \circ (1_X \Delta e_X) \circ \mathrm{pr}_X = \mathrm{pr}_X,
\end{aligned}$$

and similarly for the composition with pr_Y. □

Remark. The epi-property of η is needed in the category $\mathscr{X}$. If $\mathscr{X}$ is the category **Set**, then this requirement for η means that it is a surtransformation (all η_X's are surjections); we need this even if we consider the epi-property of η in $\mathscr{X}$ only with respect to the morphisms of $\mathscr{K}$ (i.e. morphisms in $\mathscr{X}$ which are homomorphisms in $\mathscr{K}$). This implies that for discrete algebraic structures Theorem 1 gives no better results than Theorem 2 below. But in categories of algebraic structures endowed with a continuity, like topological semigroups, convergence groups, etc., with the underlying category $\mathscr{X}$ equal to **Top, Conv, Unif**, etc., there are many examples of functors F (even reflectors) satisfying the conditions of Theorem 1 with η not a surtransformation. For concrete examples, see after Theorem 2.

Corollary 1. *Let $\mathscr{K}$ be a semi-additive category and let $F:\mathscr{K}\to\mathscr{K}$ be a covariant functor. If there is a natural transformation $\eta:1_{\mathscr{K}}\to F$ which is epi with respect to $F(\mathscr{K})$, then F preserves all finite products.*

Proof. $\mathscr{K}$ is semi-additive over itself. □

Remarks. (1) In the proof that $\mu\circ\nu=1_{FX\times FY}$ (or rather that $|\mu|\circ\nu=1_{|FX\times FY|}$) the existence of η (and its epi-property) was only used in order to show that F preserves zero-morphisms. Consequently:

If F is a covariant endofunctor of a relatively semi-additive category preserving zero-morphisms, then for finite products the morphism $|\mu|:|F(\prod X_i)|\to\prod|FX_i|$ is a retraction.

The condition that F preserves zero-morphisms cannot be left out: if $\mathscr{K}=\mathbf{Ab}$, $FX:=\mathbb{Z}\times X, Ff=1_{\mathbb{Z}}\times f$, then $\mu:\mathbb{Z}\times X\times Y\to\mathbb{Z}\times X\times\mathbb{Z}\times Y$ is given by $\mu(n,x,y)=(n,x,n,y)$(X and Y objects in **Ab**, $n\in\mathbb{Z}, x\in X, y\in Y$), hence μ is not surjective. Note that in general, if F preserves all finite products (as in the situation of the theorem) then in particular F preserves void products, that is, F preserves the zero object.

(2) In the theorem and its corollary, the epi-property for η cannot be replaced by the condition that F preserves the zero-morphisms: if $\mathscr{K}=\mathbf{Ab}$, and if for each object X in **Ab**, FX is the free abelian group over the set $|X\backslash\{0\}|$, then F preserves the zero object, but μ is not injective in general (but, by Remark 1 above, μ is a retraction). It is, in fact, easy to show that 'free algebraic structure' functors do not preserve products.

(3) The definition of a relatively semi-additive category (existence and properties of ϕ and e) cannot be weakened to the assumption that $e:1_{\mathscr{K}}\to 1_{\mathscr{K}}$ is just some natural transformation. For instance, the category of left zero semigroups (i.e., with multiplication $xy:=x$ in each of its objects) satisfies these weakened conditions: for any set X, put $e_X:=1_X$, $\phi_X:(x,y)\mapsto x:X\times X\to X$; then ϕ and e satisfy the conditions as expressed in the commutative diagrams. If X is a topological space, then e_X and ϕ_X are continuous. Stated otherwise, also the category of topological left zero semigroups satisfies the weakened conditions. Now let for each topological

left zero semigroup X, $\eta_X : X \to FX$ denote its Čech–Stone compactification (endowed with its left zero semigroup structure). Usually, $\mu : F(X \times Y) \to FX \times FY$ is not injective for Tychonov spaces X and Y.

(4) The proof of Theorem 1 is of local character: it uses only X, Y, $X \times Y$ and the images of these objects under F, together with certain morphisms between these objects. In particular, the functor $|{-}|$ has to preserve only the product under consideration, and $|{-}|$ need not reflect all isomorphisms: we need only that if $|\mu|$ is an isomorphism then so is μ. We leave it to the reader to reformulate Theorem 1 so as to apply to a fixed finite product.

(5) In the proof of Theorem 1 the axioms for ϕ were not used in full force. First, the condition $\phi \circ (\phi \times |1|) = \phi \circ (|1| \times \phi)$ was not used at all, and in addition, of the conditions $\phi \circ (|1| \Delta |e|) = |1| = \phi \circ (|e| \Delta |1|)$ the first was used for X and the second for Y. For concrete algebraic structures (such as groups, semi-groups) this means that associativity of the algebraic operation is irrelevant for Theorem 1, while X is only required to have a right unit and Y a left unit. Remark 3 above shows that the existence of some sort of unit is necessary (for remarks of similar purport in the context of **STopSGrp**, see [22, 2]).

(6) One might hope that it would be sufficient that only $F(\mathscr{K})$ is semi-additive over $\mathscr{X}$, i.e., ϕ_Z is only defined (as a morphism in $\mathscr{X}$) for objects Z of the form $Z = F(X)$ in $\mathscr{K}$. Then the first two parts of the proof of Theorem 1 are still valid (provided the other conditions of the theorem are met: the existence of $\eta : 1_{\mathscr{K}} \to F$ with each $|\eta_X|$ epic with respect to $|F(\mathscr{K})|$). In particular, $|\mu|$ is a retraction (see also Remark 1 above). But the proof of equality (6) falls through! Instead of formulating a general result about this situation, we shall give just one particular example of how to deal with this situation.

Corollary 2. *Let F be a covariant functor from* **STopSGrp** *into itself with values in* **TopSGrp**$_{\text{Haus}}$, *and assume that there exists a natural transformation $\eta : 1 \to F$ such that η_X has a dense range for each object X of* **STopSGrp**. *Then F preserves all finite products.*

Proof. (This result actually is a corollary of the proof of Theorem 1). Recall that the objects of **STopSGrp** $=: \mathscr{K}$ are assumed to have a unit element, and that morphisms preserve these unit elements (see Example 6 in Section 1). Hence $\mathscr{K}$ is 'almost' semi-additive over **Set**: all conditions of the definitions are satisfied (with $|{-}|$ the usual forgetful functor, and each $\phi_X : |X| \times |X| \to |X|$ the semi-group operation in X), except the condition that $|{-}|$ reflects isomorphisms. However, on $F(\mathscr{K})$ we shall interpret $|{-}|$ as the forgetful functor to **Haus**. Since for each object Z in $F(\mathscr{K})$ the mapping $\phi_Z : |Z| \times |Z| \to |Z|$ is continuous (i.e., ϕ_Z lifts to a morphism in **Haus**) and, moreover, each $|\eta_X|$ with X in $\mathscr{K}$ is epic with respect to $|F(\mathscr{K})|$, the first two parts of the proof of Theorem 1 can be carried out in **Haus**. In particular, ν is a morphism in **Haus** and $|\mu| \circ \nu = 1_{|FX \times FY|}$. The last part of the proof can be carried out in **Set**, showing that $\nu \circ |\mu| \circ |\eta_{X \times Y}| = |1 \eta_{X \times Y}|$. But this equality can be interpreted as an equality in **Haus** (the forgetful functor **Haus** $\to$ **Set** is faithful). Since

$|\eta_{X\times Y}|$ is epic in **Haus** it follows that $\nu \circ |\mu| = 1_{|(F(X\times Y))|}$. So $|\mu|$ is an isomorphism in **Haus**. Since the forgetful functor $\mathbf{TopSGrp}_{\mathrm{Haus}} \to \mathbf{Haus}$ reflects isomorphisms, it follows that μ is an isomorphism in $\mathbf{TopSGrp}_{\mathrm{Haus}}$, hence also in $\mathcal{K}$. □

The most general situation (and weakest result) is the following theorem. We shall say that a functor $|-|: \mathcal{K} \to \mathcal{X}$ *lifts constants* whenever for all objects X, Y in $\mathcal{K}$ the image of the set $\mathcal{K}(X, Y)$ under $|-|$ contains all constant morphisms of $\mathcal{X}(|X|, |Y|)$.

Theorem 2. *Let $\mathcal{K}$ be a category having finite products, let $\mathcal{X}$ denote either the category* **Set** *or the category* **SGrp**, *and let $|-|: \mathcal{K} \to \mathcal{X}$ be a faithful functor preserving finite products and lifting constants. If $F: \mathcal{K} \to \mathcal{K}$ is a covariant functor and $\eta: 1_{\mathcal{K}} \to F$ is a natural transformation then $|\mu|$ is injective on the image of $|\eta_{\prod X_i}|$ for each finite family $\{X_i\}$ of objects in $\mathcal{K}$.*

Proof. It is sufficient to prove the assertion for products of two factors X and Y. In view of formula (1) in the Introduction the following must be shown: if $(x, y), (x', y') \in |X| \times |Y|$ and $|\eta_X \times \eta_Y|(x, y) = |\eta_X \times \eta_Y|(x', y')$, then $|\eta_{X\times Y}|(x, y) = |\eta_{X\times Y}|(x', y')$. First, suppose that $\mathcal{X} = \mathbf{Set}$. Then for every $b \in |Y|$ we have the following commutative diagram in $\mathcal{X}$ (here c_b denotes the lifted constant morphism $X \to Y$ that has the value b in $|Y|$:

$$\begin{tikzcd}
X \times Y \arrow[r, "\eta_{X\times Y}"] & F(X \times Y) \\
X \arrow[u, "1_X \Delta c_b"] \arrow[r, "\eta_X"] & FX \arrow[u, "F(1_X \Delta c_b)"']
\end{tikzcd}$$

Together with the equality $|\eta_X|(x) = |\eta_X|(x')$ this implies

$$|\eta_{X\times Y}|(x, b) = \eta_{X\times Y}(x', b). \tag{2}$$

Similarly, the assumption $|\eta_Y|(y) = |\eta_Y|(y')$ implies for every $a \in |X|$:

$$|\eta_{X\times Y}|(a, y) = |\eta_{X\times Y}|(a, y'). \tag{3}$$

Substituting $b := y$ in (2) and $a := x'$ in (3) one gets the desired result.

In the case that $\mathcal{X} = \mathbf{SGrp}$ one obtains in a similar way the equalities (2) and (3), but now with $a := e_X$, $b := e_Y$, i.e., only for the unit elements. But as

$$(x, y) = (x, e_Y) \cdot (e_X, y), (x', y') = (x', e_Y) \cdot (e_X, y')$$

and $|\eta_{X\times Y}|$ preserves the multiplication in the semigroups, it follows easily that $|\eta_{X\times Y}|(x, y) = |\eta_{X\times Y}|(x', y')$, as desired. □

Remarks. (1) As in the proof of Theorem 1, in the above proof for a product of two factors X and Y only the existence of a right unit in X and a left unit in Y is needed.

(2) The condition that $|{-}|$ lifts constants cannot be omitted from Theorem 2. Let $\mathscr{K}$ be the category $\mathbf{Top}^G$ (see Example 8 in the Introduction). For each object $\langle X, \pi\rangle$ of $\mathbf{Top}^G$ (i.e., π is the action of G on X) let X/C_π be the orbit space of $\langle X, \pi\rangle$ and τ the trivial action of G on X/C_π. The quotient map $\eta_X : \langle X, \pi\rangle \to \langle X/C_\pi, \tau\rangle$ is a morphism in $\mathbf{Top}^G$ and it is a quotient (in fact, $\eta_X : X \to X/C_\pi$ is an open mapping). Although η is a surtransformation, μ is not injective in general: take $X = Y = G$ with $\pi(t, x) := tx$ for $t, x \in G$. Then X/C_π is a singleton, hence $(X/C_\pi) \times (Y/C_\pi)$ is a singleton. On the other hand, the orbit space of $X \times Y$ is the underlying topological space of G.

For the following corollaries, recall that if $|{-}|: \mathscr{K} \to \mathscr{X}$ is a faithful functor, then a morphism $f: X \to Y$ in $\mathscr{K}$ is said to be a *quotient* (w.r.t. $|{-}|: \mathscr{K} \to \mathscr{X}$) if $|f|$ is an epimorphism in $\mathscr{X}$ and if, in addition, $g \circ |f| \in |\mathscr{K}(X, Z)|$ for some morphism g in $\mathscr{X}$ and object Z in $\mathscr{K}$, implies $g \in |\mathscr{K}(Y, Z)|$. In our case, where $\mathscr{X}$ is either **Set** or **SGrp**, quotients are always surjective (or rather, their 'underlying' mappings in $\mathscr{X}$ are surjective; but we prefer to use adjectives like surjective, injective, etc. also for morphisms in $\mathscr{K}$).

Corollary 1. *Under the assumptions of Theorem* 2, *if η is a surtransformation, then μ is bijective for finite products. If, moreover, the faithful functor $|{-}|: \mathscr{K} \to \mathscr{X}$ reflects isomorphisms of $|F(\mathscr{K})|$, then F preserves all finite products.*

Corollary 2. *Under the assumptions of Theorem* 2, *if both η_X and η_Y are quotient and also $\eta_X \times \eta_Y$ is quotient, then $\mu : F(X \times Y) \to FX \times FY$ is an isomorphism.*

Proof. If η_X and η_Y are surjective then μ is a surjection (use formula (1)), hence a bijection. If $\eta_X \times \eta_Y$ is quotient, then (1) implies that μ is quotient. Now observe, that every bijective quotient is an isomorphism. $\square$

Applications. (1) For purely algebraic categories like **Grp**, **Ab**, R-**Mod**, **SGrp**, **BoolAlg**, **Rng**, and their full subcategories, Theorem 2 implies that each endofunctor F which admits a surtransformation (in particular: each sur-reflection) preserves finite products. As observed earlier, in this situation one cannot obtain stronger results via Theorem 1.

(2) For categories of pure continuity structures like **Top**, **Conv**, **Unif**, and their full subcategories the forgetful functor into **Set** almost never reflects isomorphisms. In these cases Theorem 2 gives only that if there is a surtransformation $\eta : 1_{\mathscr{K}} \to F$ then $\mu : F(\prod X_i) \to \prod FX_i$ is a bijection for finite products, that is, $F(\prod X_i)$ may be regarded as $\prod FX_i$ but endowed with a finer structure. This is the case for the regular and completely regular modification functors in **Top** and the precompact modification functor in **Unif** (cf. also Example 4 in the Introduction). The T_0-modification in **Top** is a quotient reflection (i.e. each η_X is quotient: it is even an open mapping) and Corollary 2 shows that it preserves all finite products. The T_1- and T_2-modifications are also quotient, but since in general $\eta_X \times \eta_Y$ need not be

quotient, (it is if η_X and η_Y are open and/or perfect maps), Corollary 2 cannot be applied, and actually, the T_1- and T_2-modifications do not preserve all finite products. There are categories of continuity structures where quotients are productive: **Unif**, **Prox** (cf. [14]), and the categories of merotopic spaces or of convergence spaces (not in **Near**, [27]). In such categories, all quotient reflections preserve finite and (Corollary 2 of Theorem 3 below) infinite products.

(3) An exception to the general statement with which 2 above begins is the category **Comp** of compact Hausdorff spaces: here the forgetful functor to **Set** reflects isomorphisms (also, surjective morphisms are quotients and quotients are productive). So if F: **Top**$\to$**Top** is a covariant functor with $F(\mathbf{Top}) \subseteq$ **Comp** and $\eta : 1 \to F$ is a surtransformation, then F preserves all finite (and, by Theorem 3 below) all infinite products. In particular, every epireflector F: **Comp**$\to$**Comp** preserves products. Example 3 from the Introduction shows that one cannot remove the epi-condition. Similar remarks can be made for the category $\mathbf{Ban}_1$ of all Banach spaces and bounded linear transformations: by the Open Mapping Theorem, bijective morphisms are isomorphisms. Thus, for example, every epireflection F: $\mathbf{Ban}_1 \to \mathbf{Ban}_1$ preserves all products. Also, in the category of standard Borel spaces and Borel mappings, every bijective morphism is an isomorphism (see [21] for references); we leave the conclusions to the reader.

(4) Corollary 2 to Theorem 1 shows the following: if $\eta_X : X \to FX$ denotes the reflection of an object X from **STopSGrp** into **CompTopSGrp** or into **CompTopGrp** (the almost peridoic, respectively strongly almost periodic compactification of X), then F preserves all finite products (for infinite products, see Theorem 4 below). For comments and references, see Example 6 in the Introduction. Here we stress the fact that our proof uses only the categorical properties of these compactifications (the proof is 'intrinsic') and make no use of (weakly) almost periodic functions. A similar method as used in the proof of Corollary 2 to Theorem 1 shows that every surreflection from **STopSGrp** into **TopSGrp** preserves finite products. A completely different application is the one, mentioned in Example 7 of the Introduction: by Theorem 1, every covariant functor F: **TopVS**$\to$**TopVS** (or $\mathcal{K} \to \mathcal{K}$, where $\mathcal{K}$ is a productive full subcategory of **TopVS**) with values in the full subcategory of Hausdorff spaces and for which there is a dense-transformation $\eta : 1_{\mathcal{K}} \to F$ preserves finite products. See also the Remark after Theorem 3 below.

(5) It is known that the category of complete convergence groups is a full reflective subcategory of the category of all convergence groups (see e.g. [23], also for earlier references to convergence groups and their completions, and [18] for later results). The natural map from a convergence group to its universal completion need not be injective, and completions need not be unique. It follows from Theorem 2 that the reflector from the category of convergence groups into the category of complete convergence groups commutes with finite products. (This result was known for abelian groups to R. Frič and V. Koutnik, but not published).

(6) If G_1 is an epireflection from $\mathbf{Top}_{3\frac{1}{2}}$ into a subcategory of **Comp** and $\prod X_i$ is pseudocompact, then $G_1(\prod X_i) = \prod G_i X_i$. Indeed, G_1 factorizes as $G_1 = G \circ F$ with

F the Čech–Stone reflector and G is an epireflector from **Comp** into itself. By Application 3 above, G preseves all products. Hence G_1 preserves every product that is preserved by F. Compare this result with [6] where in some sense a converse is obtained: if $G: \mathbf{Top}_2 \to \mathscr{A}$ and $F: \mathbf{Top}_2 \to \mathscr{B}$ are epireflections, $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathrm{Reg}\,\mathscr{A}$, where $\mathrm{Reg}\,\mathscr{A}$ is the category of all $\mathscr{A}$-regular spaces, then for all objects X, Y in $\mathrm{Reg}\,\mathscr{A}$, the equality $G(X \times Y) = GX \times GY$ implies $F(X \times Y) = FX \times FY$.

3. Infinite products

Easy examples show that in categories of discrete algebraic structures reflections do not preserve infinite products even if they preserve finite ones (according to Application 1 in Section 2). For example, take $\mathscr{K} := \mathbf{Grp}$ and for $\eta_X : X \to FX$ the quotient map of the group X onto X/X_0, where X_0 is the torsion subgroup of X. Then for $\mathbb{Z}_n$, the cyclic group of n elements, $F\mathbb{Z}_n = \{0\}$, hence $\prod_{n=1}^{\infty} F\mathbb{Z}_n = \{0\}$, but $\prod_{n=1}^{\infty} \mathbb{Z}_n$ is not a torsion group, hence $F(\prod_{zn=1}^{\infty} \mathbb{Z}_n) \neq \{0\}$. A similar example can be given for the reflection $F: \mathbf{Grp} \to \mathbf{Ab}$.

In algebraic structures, the finite products are directly determined by their factors (e.g. in **SGrp** $(x, y) = (x, e_Y) \cdot (e_X, y)$ for $(x, y) \in X \times Y$): to determine infinite products by finite ones, one needs some kind of convergence. This is done in the following theorem, which is the infinite counterpart of Theorem 2.

Theorem 3. *Let $\mathscr{K}$ be a category which admits a faithful functor $|{-}| : \mathscr{K} \to \mathbf{Top}$ and assume that $|{-}|$ preserves products. Moreover, let $F : \mathscr{K} \to \mathscr{K}$ be a covariant functor and $\eta : 1_{\mathscr{K}} \to F$ a natural transformation. If $|\mu|$ is injective on the image of $|\eta_{\prod X_i}|$ for all finite products, then $|\mu|$ is injective on the image of $|\eta_{\prod X_i}|$ for all infinite products with $|F(\prod X_i)|$ a Hausdorff space.*

Proof. Suppose that κ is an infinite ordinal number, that $\{X_\alpha\}_{\alpha \in \kappa}$ is a family of objects of $\mathscr{K}$ for which both $\prod X_\alpha$ and $\prod FX_\alpha$ exist, and that $|F(\prod X_\alpha)|$ is a Hausdorff space. Take $x, y \in |\prod X_\alpha|$ such that $|\prod \eta_\alpha|(x) = |\prod \eta_\alpha|(y)$; as in the proof of Theorem 3 we have to show that $|\eta|(x) = |\eta|(y)$ (for simplicity we write η_α instead of η_{X_α} and η instead of $\eta_{\prod X_\alpha}$). For $\beta \leq \kappa$ denote by z_β the point of $|\prod X_\alpha|$ such that

$$\mathrm{pr}_\alpha z_\beta = \begin{cases} \mathrm{pr}_\alpha y & \text{for } \alpha < \beta, \\ \mathrm{pr}_\alpha x & \text{for } \alpha \geq \beta, \end{cases}$$

(here pr_α is the projection of $|\prod X_\alpha|$ onto $|X_\alpha|$). Observe that $z_0 = x$ and $z_\kappa = y$. We shall prove by transfinite induction that $|\eta|(z_\beta) = |\eta|(z_0) = |\eta|(x)$ for all $\beta \leq \kappa$. Obviously, this is true for $\beta = 0$. Suppose our claim is true for all $\beta < \gamma$, where $0 < \gamma \leq \kappa$. If γ is isolated (i.e. $\gamma - 1$ exists) then, taking into account that $|\mu|$ is injective for the two-factor product $X_{\gamma-1} \times (\prod_{\alpha \neq \gamma-1} X_\alpha)$, one easily sees that the equality of the images of the points $z_{\gamma-1}$ and z_γ under $|\eta_{X_{\gamma-1}} \times \eta_{\prod_{\alpha \neq \gamma-1} X_\alpha}|$ implies the equality of their images under $|\eta|$. Thus, one has $|\eta|(z_{\gamma-1}) = |\eta|(z_\gamma)$. Together with the induction hypothesis it follows that $|\eta|(z_\gamma)=|\eta|(x)$. If γ is a limit then $z_\gamma=\lim_{\beta<\gamma} z_\beta$ hence $|\eta|(z_\gamma)=\lim_{\beta<\gamma}|\eta|(z_\beta)$; so by the induction hypothesis, $|\eta|(z_\gamma)=|\eta|(x)$ (observe that $F(\prod X_\alpha)$ has unique limits). This completes the proof, because for $\beta=\kappa$ the equality $|\eta|(z_\beta)=|\eta|(x)$ gives the desired result. □

Remarks. In the above proof, continuity of $|\eta|$ is needed, but only for special nets indexed over chains of length not larger than the cardinality of the index set of the product. Also, the functor $|-|:\mathscr{K}\to\mathbf{Top}$ need not preserve products in the full sense of the word: it suffices that $|\prod X_i|$ is a cartesian product endowed with a topology which is coarser than the product topology obtained when all factors are given the discrete topology (or even coarser than the chain-net coreflection of that product). So instead of **Top** one may take in Theorem 3 any convenient category of net-convergence structures; thus, in sequential structures countable products are preserved. This is formulated in the following Corollaries; here we mean by a *chain-continuity structure* a structure where convergence of chains is defined such that constant nets have their value as limit and such that each subnet of a net having a limit has the same limit. The morphisms are required to preserve the convergence.

Corollary 1. *Let $\mathscr{K}$ be a category of chain-continuity structures having a faithful functor into* **Set** *or* **SGrp** *which preserves finite products and lifts constants. If $F:\mathscr{K}\to\mathscr{K}$ is a covariant functor, has values in structures with unique limits and admits a surtransformation $\eta:1_{\mathscr{K}}\to F$, then μ is bijective on all products.*

Proof. Combine Theorem 3 (together with the Remarks above) with Theorem 2 in order to see that μ is injective. Surjectivity follows easily from equation (1) in the Introduction. □

Corollary 2. *Under the assumptions of Corollary* 1 *for $\mathscr{K}$ and F, if η is a quotient-transformation then $\mu:F(\prod X_i)\to\prod FX_i$ is an isomorphism in $\mathscr{K}$ for a product $\prod X_i$ in $\mathscr{K}$ iff $\prod \eta_{X_i}$ is quotient.*

Remarks. We leave the formulation of similar Corollaries for sequential structures and countable products to the reader. Note, that in Corollary 1, if the faithful functor from $\mathscr{K}$ into **Set** or **SGrp** reflects isomorphisms in $|F(\mathscr{K})|$, then F preserves products (if $|\mu|$ is an isomorphism, then so is μ). Also, observe that no compatibility of algebraic and continuity structures was required: we needed only continuity of $|\eta|$.

The main application of Theorem 3 and Corollary 1 lies in balanced categories (i.e. bijective morphisms are isomorphisms), and most categories with an algebraic and a continuity structure are *not* balanced. Thus, in such categories, if F is an endofunctor admitting a surtransformation $\eta:1\to F$ and F has values in Hausdorff structures, then $F(\prod X_i)$ and $\prod FX_i$ have the same underlying set, but in general the continuity structure on the former is finer than that on the latter. In categories like **Unif**, **Conv**, **STopGrp**, **TopGrp**, **TopVS**, where quotients are productive, Corollary

2 can be used. For example, the reflector from **STopGrp** into $\mathbf{TopGrp}_{\mathbf{Haus}}$ preserves all products. Another example is a modification of the example at the beginning of this Section: in the category **TopGrp** denote for an object X the torsion subgroup by X_t; then $FX := X/\bar{X}_t$ defines a reflection of **TopGrp** into $\mathbf{TopGrp}_{\mathrm{Haus}}$ (in fact, the torsion free Hausdorff groups), and F preserves all products. A similar example is obtained if one replaces X_t by X_c, the commutator subgroups of X (then one obtains the reflection into abelian Hausdorff groups). Notice that Theorem 3 and Corollary 1 are also of interest in categories of compact structures and of Banach spaces (cf. Application 3 in Section 2).

One cannot hope to obtain more than the conclusion of Theorem 3, namely, that $|\mu|$ is injective on the image of $|\eta|$. To this end, consider Example 3 of the Introduction. It is interesting (namely, in connection with the first case in Theorem 4 below) that this example can also be given within the category $\mathbf{TopGrp}_{\mathrm{Haus}}$: by [20] there exists a topological Hausdorff group S admitting no continuous endomorphisms but the obvious ones (the constant mapping with value the identity, and the identity mapping); with this object S, the procedure outlined in [9] can be performed in $\mathbf{TopGrp}_{\mathrm{Haus}}$.

Our final result is formulated in a local form in order to keep the presentation as general as possible and at the same time understandable.

Theorem 4. *Let* $|-|:\mathcal{K}\to$ **STopSGrp** *be a faithful functor which preserves products, lifts constants and reflects isomorphisms; moreover, let* $F:\mathcal{K}\to\mathcal{K}$ *be a covariant functor and* $\eta: 1_{\mathcal{K}}\to F$ *a natural transformation. If* $\{X_i\}_{i\in I}$ *is a set of objects in* $\mathcal{K}$ *then in the following cases* $\mu: F(\prod X_i)\to\prod FX_i$ *is an isomorphism*:

(1) $|F(\prod X_i)|$ *is a Hausdorff topological semigroup and* $|\eta_{\prod X_i}|$ *is surjective*;

(2) $|F(\prod X_i)|$ *is a compact Hausdorff topological semigroup and* $|\eta_{\prod X_i}|$ *maps* $|\prod X_i|$ *onto a dense subset of* $|F(\prod X_i)|$.

Proof. We have to prove that $|\mu|$ is an an isomorphism in the category **STopSGrp**. First, we shall show that $|\mu|$ is surjective. To this end, observe that for each $j\in I$ the canonical projection $p_j:\prod X_i\to X_j$ is a retraction, the diagonal product $q_j: X_j\to\prod X_i$ of 1_{X_j} and the zero-morphism $X_j\to\prod_{i\neq j}X_i$ being a section (note that $\mathcal{K}$ has zero-morphisms, obtained as liftings of the constant morphisms in **STopSGrp** that have unit elements as values). It follows that $F(p_j)$ is a retraction, so that $|F(p_j)|$ is surjective for each $j\in I$. However, $|\eta_{X_j}|\circ|p_j| = |F(p_j)|\circ|\eta_{\prod X_i}|$, and this implies that $|\eta_{X_j}|$ is surjective (has dense range, respectively) if $|\eta_{\prod X_i}|$ is surjective (has dense range, respectively). Now equation (1) in the Introduction implies that in case 1, $|\mu|$ is surjective and that in case 2, $|\mu|$ has a dense range. But in case 2, each $|FX_j|$ has a compact Hausdorff topology (being retract of $|F(\prod X_i)|$ under $|F(p_j)|$), so that $|\prod FX_i|$ has a Hausdorff topology, and therefore $|\mu|$ is a surjection in this case as well.

Next, we show that $|\mu|$ is injective. This will be sufficient for the second case, since we know already that $|\mu|$ is a surjection of compact Hausdorff structures. To prove injectivity of $|\mu|$ in case 1, we need only refer to Corollary 1 of Theorem 3 (or rather, a version of this Corollary for the given product $\prod X_i$, requiring only that each $|\eta_{X_j}|$ is surjective; cf. the proof of the Corollary). In case 2, proceed as follows. For any subset J of I, consider the following diagram in $\mathscr{K}$:

$$\begin{tikzcd}
\Pi_I X_i \arrow[r, "\eta"] \arrow[d, shift left, "p_J"] & F(\Pi_I X_i) \arrow[r, "\mu"] \arrow[d, shift left, "F(p_J)"] & \Pi_I FX_i \arrow[d, "q_J"] \\
\Pi_J X_i \arrow[r, "\eta_J"] \arrow[u, shift left, "\alpha_J"] & F(\Pi_J X_i) \arrow[r, "\mu_J"] \arrow[u, shift left, "F(\alpha_J)"] & \Pi_J FX_i
\end{tikzcd}$$

Here $\eta_J := \eta_{\prod_J X_i}$, $\eta_J := \mu_{\{X_i|i\in J\}}$, $\eta := \eta_I$, $\mu := \mu_I$, the p_J and q_J are projections, and α_J is the diagonal product of $1_{\prod_J X_i}$ with the zero-morphism $\prod_J X_i \to \prod_{I\setminus J} X_i$. Note, that for finite J the morphism $|\mu_J|$ is an isomorphism (cf. Corollary 2 to Theorem 1), so that in order to prove injectivity of $|\mu|$ it suffices to show that for $x, y \in |F(\prod X_i)|$, $x \neq y$ implies that there exists a finite subset J of I with $|F(p_J)|(x) \neq |F(p_J)|(y)$. For the proof it will be convenient to introduce the following notation:

$$|\alpha_J \circ p_J| =: w_J \quad \text{and} \quad \rho_J := |F\alpha_J \circ Fp_J| = |F\alpha_J| \circ |Fp_J|.$$

Consider any point x in $|F(\prod X_i)|$. We claim that the net $\{\rho_J x | J \in [I]^{<\omega}\}$ converges to x in $|F(\prod X_i)|$ (here $[I]^{<\omega}$ denotes the set of all finite subsets of I). Assume the contrary: there is an open nbd U of x such that the $\mathscr{F} = \{J | J \in [I]^{<\omega}, \rho_J x \notin U\}$ is cofinal in $[I]^{<\omega}$. By compactness, the set $\{\rho_J x | J \in \mathscr{F}\}$ has an accumulation point p in $|F(\prod X_i)|$. Then $p \notin U$, so p has an nbd V such that $x \notin V$. Since $p = pe$ (e the unit element in $|F(\prod X_i)|$) and the binary operation in the semigroup $|F(\prod X_i)|$ is continuous, there are nbd's V^1 of p and V_e of e such that $V^1 \cdot \bar{V}_e \subseteq V$. Continuity of $|\eta|$ implies that there is an nbd W of $\{e_i\}_{i\in I}$ in $|\prod_I X_i|$ such that $|\eta|(W) \subseteq V_e$. There is a finite subset J of I such that $w_{I\setminus J}(y) \in W$ for all $y \in \prod_I X_i$: any finite subset of I determining a basic nbd of $\{e_i\}_{i\in I}$ in $|\prod_I X_i|$, included in W, suffices; also, J can be taken large enough to guarantee that $J \in \mathscr{F}$ and $\rho_J x \in V^1$. Since $\rho_{I\setminus J}(|\eta| y) = |\eta| \circ w_{I\setminus J}(y) \in |\eta|(W) \subseteq V_e$ for all $y \in |\prod X_i|$ and $|\eta|$ has a dense range, it follows that $\rho_{I\setminus J}(z) \in \bar{V}_e$ for all $z \in |F(\prod X_i)|$. Next, notice that $y = w_J(y) \cdot w_{I\setminus J}(y)$ for all $y \in |\prod X_i|$, hence $z = \rho_J(z) \cdot \rho_{I\setminus J}(z)$ for all z in the (dense) range of $|\eta|$. By a continuity argument, this equality holds for all $z \in |F(\prod X_i)|$, which gives

$$x = \rho_J(x) \cdot \rho_{I\setminus J}(x) \in V^1 \cdot \bar{V}_e \subseteq V,$$

contradicting the choice of V. This proves our claim.

Clearly, this implies immediately that if $x, y \in |F(\prod X_i)|$, and $x \neq y$, there is $J \in [I]^{<\omega}$ with $\rho_J(x) \neq \rho_J(y)$, hence $|Fp_J|(x) \neq |Fp_J|(y)$, as desired. This completes the proof that $|\mu|$ is injective in case (2). It remains to show that $|\mu|$ is an isomorphism in case 1. We know already that it is a continuous bimorphism. That $|\mu|$ is a homeomorphism can be proved as follows.

First, notice that in $|\prod X_i|$ for each point y the net $\{w_J(y)|J\in[I]^{<\omega}\}$ converges to y. Since $|\eta|$ is a continuous surjection, it follows that in $|F(\prod X_i)|$ for each point x the net $\{\rho_J x|J\in[I]^{<\omega}\}$ converges to x. Now consider a point x and an open nbd U of x in $|F(\prod X_i)|$, and let V and V_e be nbds of x and e, respectively, such that $V\cdot V_e\subseteq U$. As in the proof above one shows that there is $J\in[I]^{<\omega}$ such that $\rho_J x\in V$ and $\rho_{I\setminus J}(z)\in V_e$ for all $z\in|F(\prod X_i)|$. Continuity of $|F\alpha_J|$ implies the existence of a nbd W of $|Fp_J|(x)$ in $|F(\prod_J X_i)|$ with $|F\alpha_J|(W)\subseteq V$. Since $|\mu_J|$ is a homeomorphism (Corollary 2 to Theorem 1), $W^1:=q_J^{-1}(|\mu_J|(W))$ is an nbd of $|\mu|(x)$ in $\prod_I|FX_i|$. Now for every point $y\in|\mu|^{-1}(W^1)$ one has $q_J|\mu|(y)\in|\mu_J|(W)$, that is, $|\mu_J|(|Fp_J|(y))\in|\mu_J|(W)$, hence $|Fp_J|(y)\in W$ and consequently $\rho_J(y)\in|F\alpha_J|(W)\subseteq V$. As before, $y=\rho_J(y)\cdot\rho_{I\setminus J}(y)$; since by the choice of J we have $\rho_{I\setminus J}(y)\in V_e$, it follows that $y\in V\cdot V_e\subseteq U$. This shows that $W^1\subseteq|\mu|(U)$ and $|\mu|$ is a homeomorphism. □

Remarks. In the last paragraph of the above proof it was observed that if $|\eta_{\prod X_i}|$ is surjective, then $\{\rho_J x|J\in[I]^{<\omega}\}$ converges to x in $|F(\prod X_i)|$. If $|F(\prod X_i)|$ has Hausdorff topology, then this can be used to give another proof of Theorem 3.

The above proof (also for case 2) can be so modified as to use only chains (then chain-compactness for chains of a certain length would be sufficient in case 2).

Finally, as in previous results, the functor $|-|$ needs only to reflect isomorphisms from $|F(\mathscr{X})|$, which is in both cases a subcategory of $\mathbf{TopSGrp}_{\mathrm{Haus}}$.

The most important applications of Theorem 4 are formulated in the following Corollaries.

Corollary 1. *The strongly almost periodic compactification* **STopSGrp** → **CompTopGrp** *and the almost periodic compactification* **STopSGrp** → **CompTopSGrp** *preserve all products.*

Proof. Both functors are reflectors, satisfying the conditions of case 2 of Theorem 4. □

Corollary 2. *Every surreflector of* **STopSGrp** *into a full subcategory of* $\mathbf{TopSGrp}_{\mathrm{Haus}}$ *preserves all products.*

Remarks. (1) The conditions on η in Theorem 4 cannot be omitted. To this end, modify Example 3 of the Introduction to one in the category $\mathbf{TopGrp}_{\mathrm{Haus}}$: let F be the reflector of this category into the subcategory $\{G^\kappa|\kappa \text{ a cardinal}\}$, where G is a strongly rigid topological Hausdorff group (cf. [20]).

(2) In the category **TopGrp**, Corollary 2 above can be improved so as to hold for dense-reflections into $\mathbf{TopGrp}_{\mathrm{Haus}}$. We shall indicate a proof of the following statement: if $F\colon \mathbf{TopGrp}\to\mathbf{TopGrp}_{\mathrm{Haus}}$ *is a covariant functor and* $\eta: 1\to F$ *is a dense-transformation, then for all products* $\mu: F(\prod X_i)\to\prod FX_i$ *is an embedding.* To prove

this, notice that η can be factorized as $1 \to^{\eta'} F' \to^{\eta''} F''$ where η' is a surtransformation and η'' is an embedding-transformation. By case 1 of Theorem 5, F' preserves all products. Thus, we need only to prove that our statement holds for the case that η is a dense-embedding transformation. With notation as in the proof of Theorem 5, let $x \in F(\prod X_i)$, $x \neq e$ and $\mu(x) = e$ (all identities are denoted e). There are disjoint nbd's U_x of x, U_e of e in $F(\prod X_i)$ and a canonical nbd V_e in $\prod X_i$ depending on some $J \in [I]^{<\omega}$ such that $\eta(V_e) \subseteq U_e$. Then $\eta^{-1}(U_x) \cap V_e = \phi$, hence $\mathrm{pr}_J \eta^{-1}(U_x) \cap \mathrm{pr}_J V_e = \phi$. But $x \in \overline{\eta(\eta^{-1}(U_x))}$ because η has a dense range, hence $e = F(\mathrm{pr}_J)(x) \in \overline{F(\mathrm{pr}_J)\eta(\eta^{-1}U_x)} = \overline{\eta_J(\mathrm{pr}_J \eta^{-1}(U_x))}$. As this set is disjoint from $\mathrm{pr}_J V_e (\eta_J$ is injective) this is a contradiction, so μ is injective. From this it follows by a straightforward argument (taking into account that $\prod FX_i$ as a product of Hausdorff groups is a regular space into which $F(\prod X_i)$ is continuously injected by μ in such a way that the dense subspace $\eta(F(\prod X_i))$ is topologically embedded) that μ is an embedding.

The following example shows that in this result μ need not be surjective: consider a sequence of topological groups $\{G_n\}_{n \in \mathbb{N}}$ such that the only continuous homomorphism from G_n to G_m for $m \neq n$ is the constant map with value the identity of G_m and such, that the only continuous endomorphisms of G_m are the constant map and the identity mapping (for the existence of such a system, consult [20] or [25]). Also, taking none of the G_n compact, the image $\tilde{G}_n$ of G_n in G_n^{SAP} is a proper subgroup of G_n^{SAP}. Now let $G := \{x \in \prod G_n^{\mathrm{SAP}} \mid \#\{n \mid \mathrm{pr}_n x \notin \tilde{G}_n\} < \omega\}$, and let F be the reflector of **TopGrp** into the epireflective hull in $\mathbf{TopGrp}_{\mathrm{Haus}}$ of $\{G\}$. Taking into account that $G_n^{\mathrm{SAP}} \subseteq G$ for each n and that G_n admits no other continuous homomorphism into G than the obvious one (coming from the canonical morphism $G_n \to G_n^{\mathrm{SAP}}$) it follows that $FG_n = G_n^{\mathrm{SAP}}$ for each n. On the other hand, $G = F(\prod G_n)$ which is a proper subset of $\prod FG_n$.

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